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INTERPHASE MASS TRANSFER BETWEEN LIQUID–LIQUID COUNTER-CURRENT FLOWS. I. VELOCITY DISTRIBUTION

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A theoretical analysis of liquid–liquid counter-current flow in laminar boundary layers with a flat interphase based on the similarity-variables method has been made. The numerical results for the velocity distribution in both phases are obtained. The dissipation energy in a boundary layer is found and the results corresponding to counter-current and co-current flows are compared. The comparison shows significant differences in the dissipation energy values in the cases of co-current and counter-current flows.

Introduction. Chemical technologies based on counter-current flows in liquid–liquid systems are widely distributed in practice. The theoretical analysis of such flows [1] demonstrates the possibility of obtaining asymptotic solutions for gas–liquid systems which are in agreement with the experimental data obtained from thermo-anemometric measurements of the velocity distribution in boundary layers.

The experience obtained in exact solution of the problem by means of numerical simulation [2] shows that it is a non-classical problem of mathematical physics which is not sufficiently discussed in the literature. A prototype of such a problem is the parabolic boundary-value problem with changing direction of time [3, 4].

Let us consider a mathematical description of a liquid flow in the boundary-layer approximation:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} , \quad v = - \int \frac{\partial u}{\partial x} dy , \tag{1}$$

where a positive velocity distribution

$$u (x, y) = u_0 (x, y) \geq 0 , \quad v (x, y) = v_0 (x, y) = - \int \frac{\partial u_0}{\partial x} dy$$

is a solution of Eq. (1), so that

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = \nu \frac{\partial^2 u_0}{\partial y^2} . \tag{2}$$

Let us consider a negative velocity distribution as a solution of Eq. (1):

$$u (x, y) = - u_0 (x, y) , \quad v (x, y) = - v_0 (x, y) ,$$

i.e., we get

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = - \nu \frac{\partial^2 u_0}{\partial y^2} . \tag{3}$$

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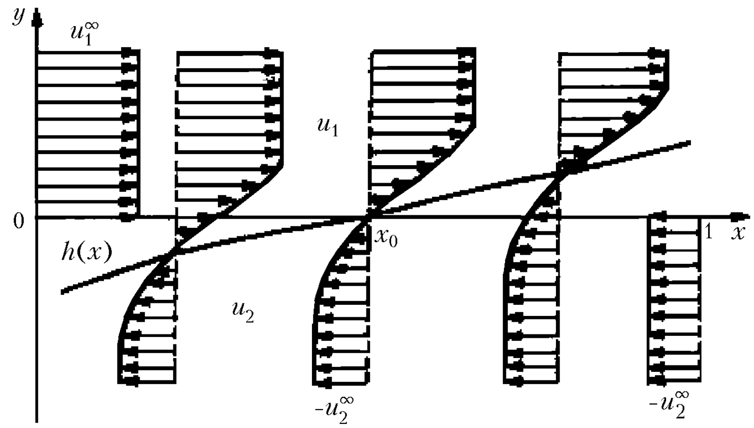

Fig. 1. Schematic of a counter-current flow.

Comparison of Eqs. (2) and (3) shows that a negative velocity distribution is not a solution of Eq. (1).

It was shown [2] that this non-classical problem can be described as consisting of several classical problems. The same approach will be used in the present work to determine velocity distribution in liquid–liquid counter-current flows with a flat interphase. At first, we consider the flows of two immiscible liquids.

Mathematical Model. The mathematical description of the counter-current flows (Fig. 1) in approximation of the boundary-layer theory has the following form:

$$u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} = \nu_i \frac{\partial^2 u_i}{\partial y^2} ,$$

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0 , \quad i = 1, 2 ;$$

$$x = 0 , \ \ y \geq 0 , \ \ u_1 = u_1^\infty ; \ \ x = l , \ \ y \leq 0 , \ \ u_2 = -u_2^\infty ;$$

$$y \to \infty , \ \ 0 \leq x \leq l , \ \ u_1 = u_1^\infty ; \qquad (4)$$

$$y \to -\infty , \ \ 0 \leq x \leq l , \ \ u_2 = -u_2^\infty ;$$

$$y = 0 , \ \ 0 < x < l , \ \ u_1 = u_2 , \ \ \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y} , \ \ v_1 = v_2 ,$$

where $y = 0$ corresponds to a flat interphase between two liquids.

Problem (4) can be presented in a dimensionless form using two different coordinate systems for the two phases, where the flows are oriented along the longitudinal coordinate. The following dimensionless variables and parameters are introduced:

$$x = lX_1 = l - lX_2 , \ \ y = \delta_1 Y_1 = -\delta_2 Y_2 , \ \ \delta_i = \sqrt{\frac{\nu_i l}{u_i^\infty}} ,$$

$$u_1 = u_1^\infty U_1 , \ \ v_1 = u_1^\infty \frac{\delta_1}{l} V_1 , \ \ u_2 = -u_2^\infty U_2 , \ \ v_2 = -u_2^\infty \frac{\delta_2}{l} V_2 ,$$

$$\theta_1 = \frac{u_2^\infty}{u_1^\infty}, \quad \theta_2 = \left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{1/2}\left(\frac{u_1^\infty}{u_2^\infty}\right)^{3/2} = \left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{1/2}\left(\frac{1}{\theta_1}\right)^{3/2}. \tag{5}$$

In the new coordinate systems, the model of counter-current flows has the following form:

$$U_i\frac{\partial U_i}{\partial X_i} + V_i\frac{\partial U_i}{\partial Y_i} = \frac{\partial^2 U_i}{\partial Y_i^2}, \quad \frac{\partial U_i}{\partial X_i} + \frac{\partial V_i}{\partial Y_i} = 0;$$

$$X_i = 0, \quad U_i = 1; \quad Y_i \to \infty, \quad U_i = 1; \tag{6}$$

$$Y_1 = Y_2 = 0, \quad U_1 = -\theta_1 U_2, \quad \theta_2\frac{\partial U_1}{\partial Y_1} = \frac{\partial U_2}{\partial Y_2}, \quad V_i = 0.$$

Method of Solution. Problem (6) cannot be solved directly because the velocities U_i change their directions in the intervals $0 \le X_i \le 1$ and $0 \le Y_i < \infty$. This non-classical problem of mathematical physics can be presented [3] as a classical one after introduction of the following similarity variables:

$$U_i = f_i', \quad V_i = \frac{1}{2\sqrt{X_i}}\left(\eta_i f_i' - f_i\right), \quad f_i = f_i(\eta_i), \quad \eta_i = \frac{Y_i}{\sqrt{X_i}}, \quad i = 1, 2. \tag{7}$$

Substitution of Eq. (7) into Eq. (6) leads to the following formulation:

$$2f_i''' + f_i f_i'' = 0;$$

$$f_i(0) = 0, \quad f_i'(\infty) = 1, \tag{8}$$

$$f_1'(0) = -\theta_1 f_2', \quad \theta_2\sqrt{\frac{X_2}{X_1}}\, f_1''(0) = f_2''(0),$$

where $X_1 + X_2 = 1$. Equation (8) has no solution in similarity variables but can be solved after introducing a new parameter $\bar{\theta}_2$ for each $X_1 \cup (0, 1)$:

$$\theta_2\sqrt{\frac{X_2}{X_1}} = \theta_2\sqrt{\frac{1-X_1}{X_1}} = \bar{\theta}_2. \tag{9}$$

Problem (8) is substituted by several separate problems for each $X_1 \cup (0, 1)$, i.e., this has a local similarity solution. In this way, the solutions of these separate problems can be obtained after introducing the function F such that

$$F(a, b) = \left|1 - f_1'(\eta_1^\infty)\right| + \left|1 - f_2'(\eta_2^\infty)\right|, \quad a = f_1'(0), \quad b = f_1''(0). \tag{10}$$

Here the values η_i^∞ should be determined in the solution.

The solution of Eq. (8) for each $X_1 \cup (0, 1)$ is obtained after finding the minimum of the function $F(a, b)$, where the following boundary problem has to be solved at each step of the minimization procedure:

$$2f_i''' + f_i f_i'' = 0, \quad f_i(0) = 0,$$

$$f_1'(0) = a, \quad f_2'(0) = -\frac{a}{\theta_1}, \quad f_1''(0) = b, \quad f_2''(0) = \bar{\theta}_2 b. \tag{11}$$

In so doing, we used the ODE 45 procedure of MATLAB 6.0.

Numerical Results. Problem (11) was solved numerically for counter-current liquid–liquid flows at different values of parameters θ_1 and θ_2. From the requirement concerning the minimum of $F(a, b)$ in (11), the boundary values of a, b, and $F(a, b)$ were determined. The results obtained for a, b, η_1^∞, and η_2^∞ are shown in Table 1, where η_i^∞ are derived from the condition $f_i'(\eta_i^\infty) = 0.99$.

The boundary conditions in Eq. (11) show that the velocity at the interphase becomes zero (as for the solid interphase), when $X_1 = X_1^0$, so that

$$f_i(0) = 0\,,\ \ f_i'(0) = 0\,,\ \ f_i''(0) = 0.33205\,, \tag{12}$$

$$\theta_2 \sqrt{\frac{1 - X_1^0}{X_1^0}} = \bar{\theta}_2 = 1\,, \tag{13}$$

which ensures the fulfillment of the condition $f_i'(\infty) = 1$.

It follows from Eq. (13) that the point X_1^0, where the velocity changes its direction at the interphase ($\eta_i = 0$), can be calculated directly at each θ_2. According to numerical simulation, at this point we have $f_1'(0) \approx 0$ and $f_2'(0) \approx 0$.

The results obtained show that there is a difference between η_1^∞ and η_2^∞ in liquid–liquid systems and $\eta^\infty = 5$ in gas (liquid)–solid systems [5] and gas–liquid systems [7] (see Table 1).

Zero-Velocity Line and Boundary-Layer Thickness. Figure 1 shows that the counter-current flow is characterized by the zero-velocity line $y = h(x)$, where

$$f_i'(\eta_i^0) = 0\,,\ \ Y_i^0 = \eta_i^0 \sqrt{X_i}\,,\ \ \eta_i^0 = \sqrt{\frac{u_i^\infty}{\nu_i}}\,\frac{h(x)}{\sqrt{x}}\,,\ \ i = 1, 2\,. \tag{14}$$

The hydrodynamic interaction between two liquids in counter-current flows is very active and differs greatly for various couples of liquids. This is the cause of the different zero-velocity lines and boundary-layer thicknesses.

The results obtained permit one to determine the zero-velocity line Y_i^0 and the laminar boundary-layer thickness $Y_i^\infty = \eta_i^\infty \sqrt{X_i}$, with the values of η_i^∞ given in Table 1.

Energy Dissipation. The energy dissipated in the laminar boundary layer [6, 7] is described for both liquids by the equation

$$e_i = \mu_i \int_0^l \left[\int_0^{\infty(-1)^{i+1}} \left(\frac{\partial u_i}{\partial y}\right)^2 dy \right] dx\,,\ \ i = 1, 2\,. \tag{15}$$

With the dimensionless variables the problem takes on the following form:

$$E_i = \int_0^l \int_0^\infty \left(\frac{\partial U_i}{\partial Y_i}\right)^2 dY_i dX_i\,, \tag{16}$$

where

$$E_i = \frac{e_i \sqrt{\nu_i/(U_i^\infty l)}}{\nu_i \rho_i (U_i^\infty)^2}\,. \tag{17}$$

Introducing the similarity variables leads to

TABLE 1. Numerical Simulation Results for Counter-Current Flows of Two Liquids

X_1	$\rho_1\mu_1/(\rho_2\mu_2)$	θ_1	θ_2	E_1	E_2	a	b	η_1^∞	η_2^∞
0.2	0.4939	0.1	22.22	0.88209	90.1	–0.0922	0.0010	21.5	2.25
0.4						–0.0947	0.0011	21.1	1.97
0.6						–0.0963	0.0012	21.0	1.55
0.8						–0.0976	0.0012	20.8	1.15
0.2	0.0034	0.3	0.355	0.7329	0.3137	0.0962	0.3275	4.70	6.71
0.4						0.1060	0.3263	4.68	8.09
0.6						0.10035	0.3269	4.71	9.06
0.8						0.08926	0.3280	4.67	10.3
0.2	0.4939	0.3	4.28	0.9598	5.05	–0.2088	0.01803	13.3	3.25
0.4						–0.2279	0.0238	12.5	3.12
0.6						–0.2424	0.02902	11.9	2.96
0.8						–0.2578	0.0355	11.5	2.66
0.2	1.9757	0.3	8.55	0.9342	17.72	–0.1807	0.1132	14.4	3.57
0.4						–0.2006	0.015836	13.6	3.30
0.6						–0.2168	0.0203	12.9	3.25
0.8						–0.2354	0.0264	12.2	3.04
0.2	20	0.3	27.22	0.8896	133.6	–0.13746	0.0045	16.9	3.92
0.4						–0.1548	0.0068	15.8	3.76
0.6						–0.1703	0.0093	15.0	3.55
0.8						–0.1899	0.0133	14.1	3.50
0.2	0.4939	0.5	1.99	0.9956	1.52	–0.2874	0.0514	10.6	3.69
0.4						–0.3170	0.0742	9.58	3.50
0.6						–0.3383	0.1002	8.85	3.28
0.8						–0.3543	0.1494	7.83	3.23
0.2	0.0034	0.7	0.1	0.5946	0.2742	0.2120	0.3110	4.39	10.1
0.4						0.18512	0.3155	4.44	11.3
0.6						0.1641	0.3194	4.48	12.2
0.8						0.1412	0.3223	4.53	13.5
0.2	0.4939	0.7	1.2	0.9095	0.9786	–0.3375	0.0989	8.94	3.92
0.4						–0.3572	0.1564	7.99	3.91
0.6						0.0938	0.3274	4.67	5.47
0.8						0.2451	0.3033	4.43	7.43
0.2	1.9757	0.7	2.4	0.9497	2.0513	–0.2918	0.05417	10.4	3.94
0.4						–0.3253	0.0832	9.48	3.92
0.6						–0.3475	0.1193	8.54	3.76
0.8						–0.3455	0.1953	7.16	3.75
0.2	20	0.7	7.64	0.9967	15.62	–0.2127	0.0191	13.0	4.10
0.4						–0.2446	0.0299	11.9	4.09
0.6						–0.2730	0.0430	11.1	4.09
0.8						–0.3079	0.0662	9.89	3.93
0.2	0.4939	1	0.703	0.7004	0.5194	–0.3429	0.2001	7.20	4.05
0.4						0.3020	0.2915	4.26	6.36
0.6						0.3543	0.2779	4.11	7.73
0.8						0.3375	0.2825	4.23	8.98
0.2	0.0034	1.5	0.032	0.6206	0.2189	0.3157	0.2885	4.23	13.1
0.4						0.2738	0.2985	4.31	14.3
0.6						0.24345	0.30505	4.37	15.4
0.8						0.2118	0.3112	4.45	16.7
0.2	0.4939	1.5	0.383	0.6276	0.3449	0.5290	0.2215	3.74	7.69
0.4						0.5126	0.2275	3.76	8.82
0.6						0.4765	0.2405	3.77	9.73
0.8						0.4260	0.2570	3.93	10.1
0.2	1.9757	1.5	0.765	0.7444	0.6799	–0.3424	0.20135	7.14	4.37
0.4						0.4982	0.2330	3.85	6.89
0.6						0.5293	0.2219	3.74	8.10
0.8						0.4959	0.2339	3.76	9.23
0.2	20	1.5	2.43	1.0027	2.48	–0.3054	0.0642	10.0	4.41
0.4						–0.3400	0.10345	8.79	4.36
0.6						–0.3541	0.1542	7.81	4.34
0.8						–0.2923	0.2582	6.31	4.48
0.2	0.0034	3	0.011	0.60	0.27	0.4398	0.2528	3.93	16.4
0.4						0.3889	0.2686	3.94	17.6
0.6						0.3518	0.2791	4.09	18.6
0.8						0.3128	0.2893	4.20	20.0
0.2	0.4939	3	0.135	0.4567	0.1289	0.7606	0.1245	3.12	11.6
0.4						0.6975	0.1535	3.25	12.4
0.6						0.6424	0.1770	3.48	13.0
0.8						0.5762	0.2040	3.54	13.8

TABLE 2. Numerical Simulation Results for Co-Current Flows of Two Liquids

$\rho_1\mu_1/(\rho_2\mu_2)$	$-\theta_1$	$-\theta_2$	a	b	η_1^∞	η_2^∞	E_1	E_2	E_1^*	E_2^*
0.4939	0.1	22.22	0.5413	0.2172	3.66	2.83	0.88209	90.17	0.1446	28.47
0.0034	0.3	0.355	0.3503	0.2793	4.08	2.63	0.7329	0.3137	0.2684	0.0233
0.4939	0.3	4.28	0.6240	0.1847	3.52	3.21	0.9598	5.05	0.1002	1.17
1.9757	0.3	8.55	0.7395	0.1340	3.21	3.12	0.9342	17.72	0.0500	2.30
20	0.3	27.22	0.8880	0.0607	2.57	3.08	0.8896	133.6	0.0096	4.40
0.4939	0.5	1.99	0.7215	0.1420	3.27	3.02	0.9956	1.52	0.0566	0.1743
0.0034	0.7	0.1	0.7174	0.1440	3.22	1.19	0.5946	0.2741	0.0583	0.0001
0.4939	0.7	1.2	0.8282	0.0913	2.89	2.62	0.9095	0.9786	0.0223	0.0283
1.9757	0.7	2.4	0.8795	0.0650	2.60	2.87	0.9497	2.0513	0.0111	0.0561
20	0.7	7.64	0.9478	0.0290	1.98	2.96	0.9967	15.62	0.0022	0.1098
0.4939	1	0.703	1.0	0	0	0	0.7004	0.5194	0	0
0.0034	1.5	0.032	1.4748	–0.3051	3.03	0.73	0.6206	0.2189	0.2015	0.0002
0.4939	1.5	0.383	1.3015	–0.1857	2.85	2.72	0.6276	0.3449	0.0784	0.0133
1.9757	1.5	0.765	1.2168	–0.1299	2.88	3.01	0.7444	0.6799	0.0394	0.0267
20	1.5	2.43	1.0967	–0.0566	2.33	3.24	1.0027	2.48	0.0078	0.0530
0.0034	3	0.011	2.9064	–1.5950	3.10	1.51	0.60	0.23	0.4130	0.0008
0.4939	3	0.135	2.2570	–0.9495	3.17	3.17	0.4567	0.1289	1.64	0.0455

$$E_i = \int_0^1 \frac{1}{\sqrt{X_i}} \left[\int_0^\infty \left(f_i'' \right)^2 d\eta_i \right] dX_i \, . \tag{18}$$

In Table 1, the dimensionless energy dissipation E_i in the boundary layer is given for the case of liquid–liquid counter-current flows at different $\theta_1 (\theta_1 = \frac{u_2^\infty}{u_1^\infty})$ and θ_2. For co-current flows, $f_i''^*$ does not depend on X_i , and the following relation is obtained for the dissipation energy:

$$E_i^* = 2 \int_0^\infty \left(f_i''^* \right)^2 d\eta_i \, , \tag{19}$$

where f_i^* is the solution of Eq. (11) under the boundary conditions for co-current flows, and

$$\theta_1^* = -\theta_1 \, , \ \ \theta_2^* = \theta_2 \, , \ \ a^* = f_1'^* \, (0) \, , \ \ b^* = f_1''^* \, (0) \, . \tag{20}$$

In Table 2, the dimensionless energy dissipation E_i in the boundary layer is presented for the case of liquid–liquid counter-current flows, which can be compared with the values E_i^* obtained for co-current flows. This table also gives the boundary conditions ($a^* = a$, $b^* = b$) and the boundary-layer thickness ($\eta_1^{\infty *} = \eta_1^\infty$, $\eta_2^{\infty *} = \eta_2^\infty$) for co-current flows in different systems at different velocities of the incoming flows. The results show that the energy dissipation for co-current flows is lower than that for counter-current flows in the same system at the same velocity.

Conclusions. The results obtained allow one to determine the velocity distribution in counter-current and co-current flows in liquid–liquid boundary layers. They open the possibility for a theoretical analysis of the heat- and mass-transfer kinetics under these conditions. The comparison between co-current and counter-current flows shows significant differences in the dissipation energy values. The boundary-layer thicknesses in two liquid–liquid laminar flows greatly differ from the same quantities in the cases of gas (liquid)–solid and gas–liquid boundary layers, where $\eta^\infty = 5$ [8].

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NOTATION

e, dissipation energy, kg·m/sec^3; *E*, dimensionless dissipation energy; *l*, length, m; *u* and *v*, velocities in *x* and *y* directions, m/sec; *x* and *y*, longitudinal and transverse coordinates, m; μ, dynamic viscosity, kg/(m·sec); ν, kinematic viscosity, m^2/sec; ρ, density, kg/m^3. Subscripts and superscripts: i = 1 and 2, liquids 1 and 2; ∞, potential flow; 0, zero-velocity line; *, co-current flow.

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